CS101 (Fall 2005) Special Topics in Computer Science

Language-Based Security

λ-Calculus Evaluation

Notation:

$e \longrightarrow v$ Expression e *evaluates to* (computes to) *value* v

$\dfrac{A_1 \quad \cdots \quad A_n}{C}$ Whenever *assumptions* A_i are true, the *conclusion* C must be true as well. If n is 0, C must be true unconditionally.

Numbers and functions are values: $\dfrac{}{n \longrightarrow n} \qquad \dfrac{}{\lambda x : t.e \longrightarrow \lambda x : t.e}$

Binary operations: $\dfrac{e_1 \longrightarrow n_1 \quad e_2 \longrightarrow n_2 \quad n = n_1 \mathit{op}\, n_2}{e_1 \mathit{op}\, e_2 \longrightarrow n}$

Function applications: $\dfrac{e_1 \longrightarrow \lambda x : t.e_1' \quad e_2 \longrightarrow v \quad e_1'[v/x] \longrightarrow v'}{e_1\ e_2 \longrightarrow v'}$

Here $e_1'[v/x]$ stands for the result of *substitution* of the value v for the variable x in expression e_1'.

λ-Calculus Evaluation Example

$$\dfrac{\dfrac{\cdots}{(\lambda f.\lambda x.f\ (f\ x))\ (\lambda x.x+1) \longrightarrow \lambda x.\Big((\lambda y.y+1)\ \big((\lambda z.z+1)\ x\big)\Big)} \quad \dfrac{}{3 \longrightarrow 3} \quad \dfrac{\dfrac{}{\lambda y.y+1 \longrightarrow \lambda y.y+1} \quad \dfrac{\dfrac{}{\lambda z.z+1 \longrightarrow \lambda z.z+1} \quad \dfrac{}{3 \longrightarrow 3} \quad \dfrac{\dfrac{}{3 \longrightarrow 3} \quad \dfrac{}{1 \longrightarrow 1} \quad 3+1=4}{3+1 \longrightarrow 4}}{(\lambda z.z+1)\ 3 \longrightarrow 4} \quad \dfrac{\dfrac{}{4 \longrightarrow 4} \quad \dfrac{}{1 \longrightarrow 1} \quad 4+1=5}{4+1 \longrightarrow 5}}{(\lambda y.y+1)\ \big((\lambda z.z+1)\ 3\big) \longrightarrow 5}}{\Big((\lambda f.\lambda x.f\ (f\ x))\ (\lambda x.x+1)\Big)\ 3 \longrightarrow 5}$$

λ-Calculus Typing Judgements

Notation:

$e \in t$	Expression e has type t
$x_1 : t_1; \cdots ; x_n : t_n \vdash e \in t$	Provided each variable x_i has type t_i, the expression e (which might have x_i in it) has type t

λ-Calculus Typing Rules

Here Γ and Δ stand for arbitrary number of *hypotheses* in a sequent.

$$\frac{}{\Gamma \vdash n \in \texttt{int}}\ (\textit{Const}) \qquad \frac{}{\Gamma; x : t; \Delta \vdash x \in t}\ (\textit{Var}) \qquad \frac{\Gamma \vdash e_1 \in \texttt{int} \quad \Gamma \vdash e_2 \in \texttt{int}}{\Gamma \vdash e_1\ \mathit{op}\ e_2 \in \texttt{int}}\ (\textit{Binop}) \qquad \frac{\Gamma \vdash e_1 \in t_1 \rightarrow t_2 \quad \Gamma \vdash e_2 \in t_1}{\Gamma \vdash (e_1\ e_2) \in t_2}\ (\textit{Apply}) \qquad \frac{\Gamma; x : t_1 \vdash e \in t_2}{\Gamma \vdash \lambda x : t_1.e \in (t_1 \rightarrow t_2)}\ (\textit{Fun})$$

λ-Calculus Typing Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{f : \texttt{int} \rightarrow \texttt{int}; x : \texttt{int} \vdash f \in \texttt{int} \rightarrow \texttt{int}}\ (V) \quad \dfrac{\dfrac{}{f : \texttt{int} \rightarrow \texttt{int}; x : \texttt{int} \vdash f \in \texttt{int} \rightarrow \texttt{int}}\ (V) \quad \dfrac{}{f : \texttt{int} \rightarrow \texttt{int}; x : \texttt{int} \vdash x \in \texttt{int}}\ (V)}{f : \texttt{int} \rightarrow \texttt{int}; x : \texttt{int} \vdash (f\ x) \in \texttt{int}}\ (A)}{f : \texttt{int} \rightarrow \texttt{int}; x : \texttt{int} \vdash f\ (f\ x) \in \texttt{int}}\ (A)}{f : \texttt{int} \rightarrow \texttt{int} \vdash \lambda x : \texttt{int}.f\ (f\ x) \in \texttt{int} \rightarrow \texttt{int}}\ (F)}{\vdash \lambda f : (\texttt{int} \rightarrow \texttt{int}).\lambda x : \texttt{int}.f\ (f\ x) \in (\texttt{int} \rightarrow \texttt{int}) \rightarrow (\texttt{int} \rightarrow \texttt{int})}\ (F) \quad \dfrac{\dfrac{\dfrac{}{x : \texttt{int} \vdash x \in \texttt{int}}\ (V) \quad \dfrac{}{x : \texttt{int} \vdash 1 \in \texttt{int}}\ (C)}{x : \texttt{int} \vdash x + 1 \in \texttt{int}}\ (B)}{\vdash \lambda x : \texttt{int}.x + 1 \in \texttt{int} \rightarrow \texttt{int}}\ (F)}{\vdash \big(\lambda f : (\texttt{int} \rightarrow \texttt{int}).\lambda x : \texttt{int}.f\ (f\ x)\big)\ (\lambda x : \texttt{int}.x + 1 \in \texttt{int} \rightarrow \texttt{int}}\ (A) \quad \dfrac{}{\vdash 3 \in \texttt{int}}\ (C)}{\vdash \Big(\big(\lambda f : (\texttt{int} \rightarrow \texttt{int}).\lambda x : \texttt{int}.f\ (f\ x)\big)\ (\lambda x : \texttt{int}.x + 1)\Big)\ 3 \in \texttt{int}}\ (A)$$