CS101C
Type Theory and Formal Methods

Lecture 9

April 30, 2003

Rules

$$\frac{}{\Gamma; x : A; \Delta \vdash x \in A}\ (\textit{type\_axiom}) \qquad \frac{\Gamma \vdash t_2 \in C \quad \Gamma; x : C \vdash t_1[x] \in T}{\Gamma \vdash t_1[t_2] \in T}\ (\textit{cut})$$

$$\frac{}{\Gamma; x : \texttt{Void}; \Delta \vdash C[x]}\ (\textit{void-E}) \qquad \frac{}{\Gamma \vdash \cdot \in \texttt{Unit}}\ (\textit{unit-T})$$

$$\frac{\Gamma; x : A \vdash t[x] \in B}{\Gamma \vdash (\lambda x.t[x]) \in (A \to B)}\ (\lambda\text{-}T) \qquad \frac{\Gamma; x : (A \to B); \Delta \vdash t_1[x] \in A \quad \Gamma; y : B; \Delta \vdash t_2[y] \in C}{\Gamma; x : (A \to B); \Delta \vdash t_2[x\, t_1[x]] \in C}\ (\to\text{-}E)$$

$$\frac{\Gamma \vdash a \in A \quad \Gamma \vdash b \in B}{\Gamma \vdash (a, b) \in (A \times B)}\ (\textit{pair-T}) \qquad \frac{\Gamma; u : A; v : B; \Delta \vdash t[(u, v)] \in C}{\Gamma; x : (A \times B); \Delta \vdash t[x] \in C}\ (\textit{pair-E})$$

$$\frac{\Gamma \vdash a \in A}{\Gamma \vdash \texttt{inl}(a) \in (A + B)}\ (\textit{inl-T}) \qquad \frac{\Gamma \vdash b \in B}{\Gamma \vdash \texttt{inr}(b) \in (A + B)}\ (\textit{inr-T})$$

$$\frac{\Gamma; u : A; \Delta[\texttt{inl}(u)] \vdash t[\texttt{inl}(u)] \in C[\texttt{inl}(u)] \quad \Gamma; v : B; \Delta[\texttt{inr}(v)] \vdash t[\texttt{inr}(v)] \in C[\texttt{inr}(v)]}{\Gamma; x : A + B; \Delta[x] \vdash t[x] \in C[x]}\ (\textit{union-E})$$

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$$\frac{}{\Gamma;\ A;\Delta \vdash A}\ (\textit{type\_axiom}) \qquad \frac{\Gamma \vdash C \quad \Gamma;\ C \vdash T}{\Gamma \vdash T}\ (\textit{cut})$$

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$$\frac{\Gamma;\ A \vdash B}{\Gamma \vdash (A \to B)}\ (\lambda\textit{-T}) \qquad \frac{\Gamma;\ (A \to B);\Delta \vdash A \quad \Gamma;\ B;\Delta \vdash C}{\Gamma;\ (A \to B);\Delta \vdash C}\ (\to\textit{-E})$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash (A \times B)}\ (\textit{pair-T}) \qquad \frac{\Gamma;\ A;\ B;\Delta \vdash C}{\Gamma;\ (A \times B);\Delta \vdash C}\ (\textit{pair-E})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash (A + B)}\ (\textit{inl-T}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash (A + B)}\ (\textit{inr-T})$$

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$$\frac{}{\Gamma; \texttt{Void}; \Delta \vdash C[x]}\ (\mathit{void\text{-}E}) \qquad \frac{}{\Gamma \vdash \texttt{Unit}}\ (\mathit{unit\text{-}T})$$

$$\frac{\Gamma; A \vdash B}{\Gamma \vdash (A \rightarrow B)}\ (\lambda\text{-}T) \qquad \frac{\Gamma; (A \rightarrow B); \Delta \vdash A \quad \Gamma; B; \Delta \vdash C}{\Gamma; (A \rightarrow B); \Delta \vdash C}\ (\rightarrow\text{-}E)$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash (A \times B)}\ (\mathit{pair\text{-}T}) \qquad \frac{\Gamma; A; B; \Delta \vdash C}{\Gamma; (A \times B); \Delta \vdash C}\ (\mathit{pair\text{-}E})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash (A + B)}\ (\mathit{inl\text{-}T}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash (A + B)}\ (\mathit{inr\text{-}T})$$

$$\frac{\Gamma; A; \Delta \vdash C \quad \Gamma; B; \Delta \vdash C}{\Gamma; A + B; \Delta \vdash C}\ (\mathit{union\text{-}E})$$

Propositions-as-Types

For every intuitionistic proposition, we construct a type of all the possible evidence for that proposition. The proposition is true when the type is non-empty.

Proposition	Type
$\bot$	$\mathtt{Void}$
$\top$	$\mathtt{Unit}$
$A \Rightarrow B$	$A \rightarrow B$
$A \wedge B$	$A \times B$
$A \vee B$	$A + B$

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???	$x : A \rightarrow B[x]$
???	$x : A \times B[x]$