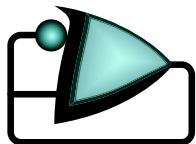


CS101C  
**Type Theory  
and Formal Methods**

Lecture 2

April 2, 2003



# Soundness and Completeness

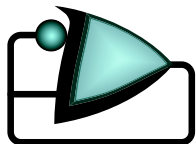
Recall: A *proof* is a sequence of *sentences*, where each one follows from previous statements in the proof using a *rule of inference* (possibly *axiom*).

Must any sentence we can prove be actually *true*?

Suppose we know what the *semantics* of the language is, we know what it means for a sentence to be true.

A proof system is called *sound* if everything we can prove is true.

A proof system is called *complete* if any true sentence can be proven.



# Inference Rules for Classical Logic — Reminder

$$\frac{}{\Gamma; A; \Delta \vdash A} \text{ (Axiom)} \quad \frac{\Gamma; A; A; \Delta \vdash C}{\Gamma; A; \Delta \vdash C} \text{ (Copy)} \quad \frac{\Gamma; \Delta \vdash C}{\Gamma; A; \Delta \vdash C} \text{ (Weakening)}$$

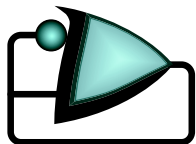
$$\frac{\Gamma; \Delta; \Gamma'; \Delta' \vdash C}{\Gamma; \Gamma'; \Delta; \Delta' \vdash C} \text{ (Swap)} \quad \frac{}{\Gamma \vdash \top} \text{ (True-intro)} \quad \frac{}{\Gamma; \perp; \Delta \vdash C} \text{ (False-elim)}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \text{ (Or-intro-1)} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \text{ (Or-intro-2)} \quad \frac{\Gamma; A; \Delta \vdash C \quad \Gamma; B; \Delta \vdash C}{\Gamma; A \vee B; \Delta \vdash C} \text{ (Or-elim)}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \text{ (And-intro)} \quad \frac{\Gamma; A; B; \Delta \vdash C}{\Gamma; A \wedge B; \Delta \vdash C} \text{ (And-elim)}$$

$$\frac{\Gamma; A \vdash B}{\Gamma \vdash A \Rightarrow B} \text{ (Imp-intro)} \quad \frac{\Gamma; \Delta \vdash A \quad \Gamma; B; \Delta \vdash C}{\Gamma; A \Rightarrow B; \Delta \vdash C} \text{ (Imp-elim)}$$

$$\frac{\Gamma; A \vdash \perp}{\Gamma \vdash \neg A} \text{ (Not-intro)} \quad \frac{\Gamma; \Delta \vdash A}{\Gamma; \neg A; \Delta \vdash C} \text{ (Not-elim)} \quad \frac{\Gamma; \neg A \vdash \perp}{\Gamma \vdash A} \text{ (Proof by contradiction)}$$

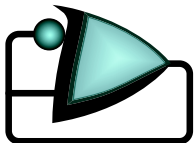


# Semantics of Classical Logic

A formula is called *tautology*, if for any assignment of truth values ( $\top/\perp$ ) for variables, it evaluates to  $\top$ .

Examples:  $A \vee \neg A$ ,  $A \Rightarrow A$ ,  $\top$ ,  $A \Rightarrow (B \Rightarrow A)$ .

A sequent  $A_1; \dots; A_n \vdash C$  is true, if  $(A_1 \wedge \dots \wedge A_n) \Rightarrow C$  is a tautology.



# Inference Rules for Classical Logic — Soundness

$$\frac{}{\Gamma; A; \Delta \vdash A} \text{ (Axiom)} \quad \frac{\Gamma; A; A; \Delta \vdash C}{\Gamma; A; \Delta \vdash C} \text{ (Copy)} \quad \frac{\Gamma; \Delta \vdash C}{\Gamma; A; \Delta \vdash C} \text{ (Weakening)}$$

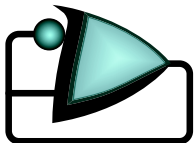
$$\frac{\Gamma; \Delta; \Gamma'; \Delta' \vdash C}{\Gamma; \Gamma'; \Delta; \Delta' \vdash C} \text{ (Swap)} \quad \frac{}{\Gamma \vdash \top} \text{ (True-intro)} \quad \frac{}{\Gamma; \perp; \Delta \vdash C} \text{ (False-elim)}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \text{ (Or-intro-1)} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \text{ (Or-intro-2)} \quad \frac{\Gamma; A; \Delta \vdash C \quad \Gamma; B; \Delta \vdash C}{\Gamma; A \vee B; \Delta \vdash C} \text{ (Or-elim)}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \text{ (And-intro)} \quad \frac{\Gamma; A; B; \Delta \vdash C}{\Gamma; A \wedge B; \Delta \vdash C} \text{ (And-elim)}$$

$$\frac{\Gamma; A \vdash B}{\Gamma \vdash A \Rightarrow B} \text{ (Imp-intro)} \quad \frac{\Gamma; \Delta \vdash A \quad \Gamma; B; \Delta \vdash C}{\Gamma; A \Rightarrow B; \Delta \vdash C} \text{ (Imp-elim)}$$

$$\frac{\Gamma; A \vdash \perp}{\Gamma \vdash \neg A} \text{ (Not-intro)} \quad \frac{\Gamma; \Delta \vdash A}{\Gamma; \neg A; \Delta \vdash C} \text{ (Not-elim)} \quad \frac{\Gamma; \neg A \vdash \perp}{\Gamma \vdash A} \text{ (Proof by contradiction)}$$

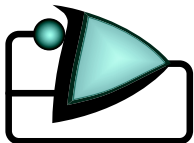




# Homework 1

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- **Part I:** Make sure you can access UGCS and run MetaPRL there.  
Due: Monday, 2PM
- **Part II:** Write 3 formal proofs (on paper)  
Due: Monday, 2:55PM



# Evidence Semantics

Informally: Instead of requiring that a formula is  $\top$ , we require that know how to get “evidence” of it being true.

$\top$  self-evident

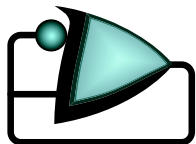
$\perp$  will never be able to get any evidence

$A \wedge B$  evidence of  $A$  and evidence of  $B$

$A \vee B$  evidence of either  $A$  or  $B$  (with information on which one)

$A \Rightarrow B$  if somebody give us evidence of  $A$ , we’ll be able to get evidence of  $B$

$A_1; \dots; A_n \vdash C$  if we’ll get evidence for all  $A_i$ , we’ll be able to come up with evidence for  $C$



# Soundness in Evidence Semantics

$$\frac{}{\Gamma; A; \Delta \vdash A} \text{ (Axiom)} \quad \frac{\Gamma; A; A; \Delta \vdash C}{\Gamma; A; \Delta \vdash C} \text{ (Copy)} \quad \frac{\Gamma; \Delta \vdash C}{\Gamma; A; \Delta \vdash C} \text{ (Weakening)}$$

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$$\frac{\Gamma; A \vdash \perp}{\Gamma \vdash \neg A} \text{ (Not-intro)} \quad \frac{\Gamma; \Delta \vdash A}{\Gamma; \neg A; \Delta \vdash C} \text{ (Not-elim)} \quad \frac{\Gamma; \neg A \vdash \perp}{\Gamma \vdash A} \text{ (Proof by contradiction)}$$

