CS101C

Type Theory and Formal Methods

Lecture 2

April 2, 2003

Soundness and Completeness

Recall: A *proof* is a sequence of *sentences*, where each one follows from previous statements in the proof using a *rule of inference* (possibly *axiom*).

Must any sentence we can prove be actually *true*?

Suppose we know what the *semantics* of the language is, we know what it means for a sentence to be true.

A proof system is called *sound* if everything we can prove is true.

A proof system is called *complete* if any true sentence can be proven.

Inference Rules for Classical Logic — Reminder

$$\frac{}{\Gamma;\ A;\ \Delta \vdash A}\ (\textit{Axiom}) \qquad \frac{\Gamma;\ A;\ A;\ \Delta \vdash C}{\Gamma;\ A;\ \Delta \vdash C}\ (\textit{Copy}) \qquad \frac{\Gamma;\ \Delta \vdash C}{\Gamma;\ A;\ \Delta \vdash C}\ (\textit{Weakening})$$

$$\frac{\Gamma;\ \Delta;\ \Gamma';\Delta' \vdash C}{\Gamma;\ \Gamma';\Delta;\ \Delta' \vdash C}\ (\textit{Swap}) \qquad \frac{}{\Gamma \vdash \top}\ (\textit{True-intro}) \qquad \frac{}{\Gamma;\ \bot;\ \Delta \vdash C}\ (\textit{False-elim})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\textit{Or-intro-1}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\textit{Or-intro-2}) \qquad \frac{\Gamma;\ A;\ \Delta \vdash C \quad \Gamma;\ B;\ \Delta \vdash C}{\Gamma;\ A \vee B;\ \Delta \vdash C}\ (\textit{Or-elim})$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\textit{And-intro}) \qquad \frac{\Gamma;\ A;\ B;\ \Delta \vdash C}{\Gamma;\ A \wedge B;\ \Delta \vdash C}\ (\textit{And-elim})$$

$$\frac{\Gamma;\ A \vdash B}{\Gamma \vdash A \Rightarrow B}\ (\textit{Imp-intro}) \qquad \frac{\Gamma;\ \Delta \vdash A \quad \Gamma;\ B;\ \Delta \vdash C}{\Gamma;\ A \Rightarrow B;\ \Delta \vdash C}\ (\textit{Imp-elim})$$

$$\frac{\Gamma;\ A \vdash \bot}{\Gamma \vdash \neg A}\ (\textit{Not-intro}) \qquad \frac{\Gamma;\ \Delta \vdash A}{\Gamma;\ \neg A;\ \Delta \vdash C}\ (\textit{Not-elim}) \qquad \frac{\Gamma;\ \neg A \vdash \bot}{\Gamma \vdash A}\ (\textit{Proof by contradiction})$$

Semantics of Classical Logic

A formula is called *tautology*, if for any assignment of truth values ($\top/\bot$) for variables, it evaluates to $\top$.

Examples: $A \vee \neg A$, $A \Rightarrow A$, $\top$, $A \Rightarrow (B \Rightarrow A)$.

A sequent $A_1; \ldots; A_n \vdash C$ is true, if $(A_1 \wedge \ldots \wedge A_n) \Rightarrow C$ is a tautology.

Inference Rules for Classical Logic — Soundness

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$$\frac{\Gamma;\ A \vdash \bot}{\Gamma \vdash \neg A}\ (\textit{Not-intro}) \qquad \frac{\Gamma;\ \Delta \vdash A}{\Gamma;\ \neg A;\ \Delta \vdash C}\ (\textit{Not-elim}) \qquad \frac{\Gamma;\ \neg A \vdash \bot}{\Gamma \vdash A}\ (\textit{Proof by contradiction})$$

Homework 1

- **Part I**: Make sure you can access UGCS and run MetaPRL there.
  Due: Monday, 2PM
- **Part II**: Write 3 formal proofs (on paper)
  Due: Monday, 2:55PM

Evidence Semantics

Informally: Instead of requiring that a formula is $\top$, we require that know how to get "evidence" of it being true.

$\top$	self-evident
$\bot$	will never be able to get any evidence
$A \wedge B$	evidence of A and evidence of B
$A \vee B$	evidence of either A or B (with information on which one)
$A \Rightarrow B$	if somebody give us evidence of A, we'll be able to get evidence of B
$A_1; \ldots; A_n \vdash C$	if we'll get evidence for all A_i, we'll be able to come up with evidence for C

Soundness in Evidence Semantics

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